ISRAEL JOURNAL OF MATHEMATICS **128** (2002), 77–92

ON CONTRACTIVELY COMPLEMENTED SUBSPACES OF SEPARABLE L_1-PREDUALS

BY

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ABSTRACT

It is shown that for an L_1-predual space X and a countable linearly independent subset of $\text{ext}(B_{X^*})$ whose norm-closed linear span Y in X^* is w^*-closed, there exists a w^*-continuous contractive projection from X^* onto Y. This result combined with those of Pelczynski and Bourgain yields a simple proof of the Lazar–Lindenstrauss theorem that every separable L_1-predual with non-separable dual contains a contractively complemented subspace isometric to $C(\Delta)$, the Banach space of functions continuous on the Cantor discontinuum Δ.

It is further shown that if X^* is isometric to ℓ_1 and (e_n^*) is a basis for X^* isometrically equivalent to the usual ℓ_1-basis, then there exists a w^*-convergent subsequence $(e_{m_n}^*)$ of (e_n^*) such that the closed linear subspace of X^* generated by the sequence $(e_{m_{2n}}^* - e_{m_{2n-1}}^*)$ is the range of a w^*-continuous contractive projection in X^*. This yields a new proof of Zippin's result that c_0 is isometric to a contractively complemented subspace of X.

1. Introduction

A Banach space X is said to be an L_1-predual provided its dual X^* is isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ). Perhaps the most natural example of an L_1-predual is $C(K)$, the Banach space of real-valued functions continuous on the compact Hausdorff space K, under the supremum norm. L_1-preduals were the subject of an extensive study in the late 1960's and early 1970's. For

Received September 15, 2000

a detailed survey of results on L_1-preduals we refer to [13]. For the connection between L_1-preduals and infinite-dimensional convexity we refer to the recent survey article [8].

For some time it was thought that every L_1-predual is isomorphic to a $C(K)$-space for suitable K, but the example given by Benyamini and Lindenstrauss [3] disproved this. The present paper is concerned with the existence of subspaces of a separable infinite-dimensional L_1-predual X, isometric to $C(K)$-spaces. It was proven by Zippin [27] that X contains a contractively complemented subspace isometric to c_0. When X^* is non-separable, Lazar and Lindenstrauss [16] proved that X contains a contractively complemented subspace isometric to $C(\Delta)$, where Δ denotes the Cantor discontinuum. These results complement each other in the sense that neither of them implies the other.

In the present paper we demonstrate a unified approach towards these results. Our method consists of establishing Theorem 3.2 which describes a technique for constructing a strictly increasing sequence (V_n) of finite-dimensional subspaces of X^*, where each V_n is isometric to some $\ell_1^{k_n}$, for which there exists an almost commuting sequence (P_n) of w^*-continuous contractive projections in X^* such that $\operatorname{Im} P_n = V_n$, $n \in \mathbb{N}$. We recall here that a sequence (P_n) of uniformly bounded projections in a Banach space E is said to be **almost commuting**, if $\lim_k \sup_{n \geq k} \|P_k P_n - P_k\| = 0$. The proof of Theorem 3.2 is an elementary application of the principle of local reflexivity [17], [10]. We apply Theorem 3.2 in order to provide an alternative proof for the following result

THEOREM 1.1: *Let X be an L_1-predual and let K be a countable subset of $\operatorname{ext}(B_{X^*})$ such that $K \cap (-K) = \emptyset$. Suppose that the norm-closed linear span Y of K is w^*-closed in X^*. Then there exists a w^*-continuous contractive projection from X^* onto Y.*

We remark that Theorem 1.1 is a consequence of the following result formulated by Lazar and Lindenstrauss as Corollary 1 in [16]:

Suppose that X is a Banach space so that X^* is isometric to $L_1(\mu)$. Let F be a face of B_{X^*} and denote by H the convex hull of $F \cup (-F)$. Assume that H is w^*-closed and metrizable. Then, there exists a w^*-continuous, affine, symmetric retraction of B_{X^*} onto *H.*

Evidently, this result yields Theorem 1.1. However, the authors of [16] do not offer a detailed proof of this result and moreover, via their preceding discussion, seem to require that the face F be w^*-closed. Note that if F is w^*-closed, so is H but the converse is not true in general. We also note that the proof of the aforementioned result that appears in [13] is false. Specifically, the map ϕ defined in the proof of the Corollary on page 224 of [13] is not convex.

Theorem 1.1 has applications in the study of ℓ_1-preduals [1], [9]. As a consequence of Theorem 1.1 we obtain

COROLLARY 1.2: *Let X be a separable L_1-predual and let K be a countable, w^*-compact subset of $\operatorname{ext}(B_{X^*})$ such that $K \cap (-K) = \emptyset$. Then there exists a contractively complemented subspace of X isometric to $C(K)$.*

This corollary combined with the results of Pelczynski [22] and Bourgain [4] yields the next

COROLLARY 1.3: *Let X be a separable L_1-predual such that X^* is non-separable. Then there exists a contractively complemented subspace of X isometric to $C(\Delta)$.*

This result was obtained in [16] with a different method. Their proof is based on a remarkable affine version of Michael's selection theorem [19] and makes use of a non-trivial result established in [14].

Another application of Theorem 3.2 is the following

THEOREM 1.4: *Let X be a Banach space such that X^* is isometric to ℓ_1, and let (e_n^*) be a basis for X^* isometrically equivalent to the usual ℓ_1-basis. Then there exists a w^*-convergent subsequence $(e_{m_n}^*)$ of (e_n^*) such that the subspace generated by the sequence $(e_{m_{2n}}^* - e_{m_{2n-1}}^*)$ is the range of a w^*-continuous contractive projection in X^*.*

Theorem 1.4 combined with Corollary 1.3 yields an alternative proof of Zippin's result [27]

COROLLARY 1.5: *Every separable infinite-dimensional L_1-predual X contains a contractively complemented subspace isometric to c_0.*

2. Preliminaries

We shall make use of standard Banach space facts and terminology as may be found in [18]. In this section we review some of the necessary concepts. All Banach spaces under consideration will be over the field of real numbers. By the term **subspace** of a Banach space X we shall mean a closed linear subspace. We let B_X stand for the closed unit ball of X, while X^* denotes its topological dual. A subspace Y of X is said to be **complemented** if it is the range of a bounded linear projection $P: X \to X$. When $\|P\| = 1$, Y is a **contractively** complemented subspace of X.

ℓ_1 denotes the Banach space of absolutely summable sequences under the norm given by the sum of the absolute values of the coordinates. The usual ℓ_1-basis is the Schauder basis of ℓ_1 consisting of sequences having exactly one coordinate equal to 1 and vanishing at the rest of the coordinates. ℓ_1^k, where $k \in \mathbb{N}$, is the k-dimensional subspace of ℓ_1 spanned by the first k members of the usual ℓ_1-basis. A sequence (x_n) in some Banach space is **isometrically equivalent** to the usual ℓ_1-basis, if $\|\sum_{i=1}^n a_i x_i\| = \sum_{i=1}^n |a_i|$, for all $n \in \mathbb{N}$ and scalar sequences $(a_i)_{i=1}^n$. A finite sequence $(x_i)_{i=1}^k$ in some Banach space is isometrically equivalent to the usual ℓ_1^k-basis, if $\|\sum_{i=1}^k a_i x_i\| = \sum_{i=1}^k |a_i|$, for all scalar sequences $(a_i)_{i=1}^k$.

c_0 stands for the Banach space of null sequences under the norm given by the supremum of the absolute values of the coordinates. ℓ_∞^n denotes the Banach space $\mathbb{R}^n$ under the norm given by the maximum of the absolute values of the coordinates.

Given a measure space (Ω, Σ, μ) with μ positive, $L_1(\mu)$ denotes the Banach space of equivalence classes of absolutely integrable functions on Ω under the norm $\|f\| = \int_\Omega |f| d\mu$. $L_\infty(\mu)$ denotes the Banach space of equivalence classes of essentially bounded Σ-measurable functions on Ω under the norm $\operatorname{ess\,sup}_{\omega \in \Omega} |f(\omega)|$.

An L_1-**predual** is a Banach space X such that X^* is isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ). According to a result of Pelczynski [21], Proposition 1.3, there exists another measure space (Ω', Σ', ν) with $L_1(\nu)$ isometric to $L_1(\mu)$ and such that $L_1(\nu)^*$ is canonically isometric to $L_\infty(\nu)$. Thus in the sequel, by an L_1-predual we shall mean a Banach space X with X^* isometric to some $L_1(\mu)$ such that $L_1(\mu)^*$ is canonically isometric to $L_\infty(\mu)$.

Given a linear topological space V and $A \subset V$, we let $\operatorname{co}(A)$ denote the convex hull of A. Let now K be a convex subset of V. A point $x \in K$ is called an **extreme point**, if whenever y, z are in K and $x = ay + (1-a)z$ for some $0 < a < 1$, then $x = y = z$. We let $\operatorname{ext}(K)$ denote the set of extreme points of K. It is well known that for an L_1-predual space X with X^* isometric to $L_1(\mu)$, $\operatorname{ext}(B_{X^*})$ consists precisely of functions of the form $\sigma\chi_A/\mu(A)$, where $\sigma \in \{-1, 1\}$, A is an atom with $0 < \mu(A) < \infty$ and χ_A stands for the indicator function of A.

We next recall the important principle of local reflexivity [17], [10] (cf. also [26]).

THEOREM 2.1: *Let X be a Banach space and let $E \subset X^{**}$ and $F \subset X^*$ be finite-dimensional subspaces. Given $\epsilon > 0$, there exists an invertible linear operator $T\colon E \to X$ such that $\|T\|\|T^{-1}|_{TE}\| \leq 1+\epsilon$, $T|E \cap X = id_{E\cap X}$, and $f(Te) = e(f)$*

for all $f \in F$ and $e \in E$.

3. A construction of w^*-continuous contractive projections

This section is devoted to the proof of Theorem 3.2 which provides a method of constructing w^*-continuous contractive projections onto certain finite-dimensional subspaces of $X^* = L_1(\mu)$, isometric to ℓ_1^k. Repeated applications of Theorem 3.2 will in turn enable us to construct a sequence (P_n) of almost commuting w^*-continuous contractive projections in X^* such that $(\operatorname{Im} P_n)$ is strictly increasing and $\operatorname{Im} P_n$ is isometric to some $\ell_1^{k_n}$, for all $n \in \mathbb{N}$. In order to construct w^*-continuous projections onto subspaces of X^* isometric to ℓ_1, we shall make use of the following

PROPOSITION 3.1: *Let X be a Banach space and let Y be a w^*-closed subspace of X^*. Assume that there exists a net $\{Y_\lambda\}_{\lambda\in\Lambda}$ of w^*-closed subspaces of Y with $Y_{\lambda_1} \subset Y_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$ in Λ, and such that $\bigcup_{\lambda\in\Lambda} Y_\lambda$ is norm-dense in Y. Assume further that each Y_λ is the range of a w^*- continuous projection P_λ in X^*, so that $\sup_\lambda \|P_\lambda\| \leq M < \infty$, and $\lim_\lambda \sup_{\lambda\leq\mu} \|P_\lambda P_\mu - P_\lambda\| = 0$. Then there exists a w^*-continuous projection P from X^* onto Y with $\|P\| \leq M$.*

Proof: B_Y is w^*-compact. By Tychonoff's theorem we infer that $K = \prod_{x^*\in B_{X^*}} MB_Y$ is compact when endowed with the cartesian topology. We can now identify $\{P_\lambda\}_{\lambda\in\Lambda}$ with a net in K to obtain a sub-net $\{P_{\lambda'}\}_{\lambda'\in\Lambda'}$ of $\{P_\lambda\}_{\lambda\in\Lambda}$ which converges to an element P of K. Since Y is w^*-closed in X^*, P induces a bounded linear operator from X^* into Y, which we still denote by P. Clearly $Px^* = w^* - \lim_{\lambda'\in\Lambda'} P_{\lambda'}x^*$, for all $x^* \in X^*$ and thus $\|P\| \leq M$. Our assumptions yield that $Px^* = x^*$, for all $x^* \in \bigcup_{\lambda\in\Lambda} Y_\lambda$ and hence P is a projection onto Y.

We next demonstrate that P is w^*-continuous. By a classical result [12] it suffices to show that for every net (x_ν^*) in B_{X^*} such that $w^* - \lim_\nu x_\nu^* = \mathbf{0}$, we have that $w^* - \lim_\nu Px_\nu^* = \mathbf{0}$. Note that $\|Px_\nu^*\| \leq M$, for all ν, and let $y^* \in MB_Y$ be any w^*-cluster point of $(Px_\nu^*)_\nu$. We will show that $y^* = \mathbf{0}$. To this end set $\delta_\lambda = \sup_{\lambda\leq\mu} \|P_\lambda P_\mu - P_\lambda\|$. Then $\|P_\lambda P - P_\lambda\| \leq \delta_\lambda$, as P_λ is w^*-continuous. It follows that $\|P_\lambda P x_\nu^* - P_\lambda x_\nu^*\| \leq \delta_\lambda$, for all λ and ν, and thus as P_λ is w^*-continuous, we obtain that $\|P_\lambda y^*\| \leq \delta_\lambda$, for all $\lambda \in \Lambda$. Hence $Py^* = \mathbf{0}$. Because $y^* \in Y$ and P is a projection onto Y, we deduce that $y^* = \mathbf{0}$, completing the proof of the assertion. ∎

We next pass to the key result which is related to Lemma 3.1 and Corollary 3.2 of [10].

THEOREM 3.2: *Let X be an L_1-predual and let V be a subspace of X^* isometric to ℓ_1^k. Let $(\delta_i)_{i=1}^n$ be a finite sequence of positive scalars and assume that there exist w^*-continuous linear operators T_i: $X^* \to V$, $\|T_i\| \le 1$, $i \le n$, as well as linear operators R_i: $V \to V$, $\|R_i\| \le 1$, $i \le n$, so that $\|R_i T_n - T_i\| < \delta_i$ for all $i \le n$. Assume further that there exist collections of vectors $(f_j)_{j=1}^q \subset \mathrm{ext}(B_{X^*})$ and $(v_j)_{j=1}^q \subset B_V$, with $(f_j)_{j=1}^q$ linearly independent, such that $\|R_i v_j - T_i f_j\| < \delta_i$ for all $i \le n$ and $j \le q$. Then there exists a w^*-continuous linear operator T: $X^* \to V$, $\|T\| \le 1$, such that $\|R_i T - T_i\| < \delta_i$ for all $i \le n$, and $Tf_j = v_j$ for all $j \le q$.*

The proof of this result will follow after establishing the next

PROPOSITION 3.3: *Under the hypothesis of Theorem 3.2, for every $\epsilon > 0$ there exists a w^*-continuous linear operator S: $X^* \to V$, $\|S\| \le 1$, such that $\|R_i S - T_i\| < \delta_i + \epsilon$ and $\|Sf_j - v_j\| < \epsilon$ for all $i \le n$ and $j \le q$.*

Proof: Let $(e_l)_{l=1}^k$ be a basis for V isometrically equivalent to the usual ℓ_1^k-basis. Since T_i is w^*-continuous and $\|T_i\| \le 1$, there exist vectors $(z_{i,l})_{l=1}^k$ in B_X such that $T_i x^* = \sum_{l=1}^k x^*(z_{i,l}) e_l$ for all $x^* \in X^*$. There also exist scalars $(a_{i,l,s})$, $i \le n$, $l \le k$, $s \le k$, such that $R_i e_l = \sum_{s=1}^k a_{i,l,s} e_s$ for all $i \le n$ and $l \le k$. Finally, there exist scalars $(v_{j,l})$, $j \le q$, $l \le k$, such that $v_j = \sum_{l=1}^k v_{j,l} e_l$, $j \le q$. Observe that for $x^* \in X^*$ and $i \le n$ we have

$$\sum_{s=1}^k \left(\sum_{l=1}^k a_{i,l,s} x^*(z_{n,l}) \right) e_s = \sum_{l=1}^k x^*(z_{n,l}) \sum_{s=1}^k a_{i,l,s} e_s = R_i T_n x^*.$$

Hence, $\sum_{s=1}^k \left| \left(\sum_{l=1}^k a_{i,l,s} x^*(z_{n,l}) \right) - x^*(z_{i,s}) \right| = \|R_i T_n x^* - T_i x^*\| < \delta_i$, for all $x^* \in B_{X^*}$ and $i \le n$. Thus

$$\left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{i,l,s} z_{n,l} \right) - z_{i,s} \right] \right\| < \delta_i, \tag{1}$$

for all choices of signs $(\rho_s)_{s=1}^k$ and all $i \le n$.

Similarly, $R_i v_j = \sum_{s=1}^k (\sum_{l=1}^k a_{i,l,s} v_{j,l}) e_s$ for all $i \le n$, $j \le q$, and thus

$$\sum_{s=1}^k \left| \left(\sum_{l=1}^k a_{i,l,s} v_{j,l} \right) - f_j(z_{i,s}) \right| = \|R_i v_j - T_i f_j\| < \delta_i, \quad j \le q, \quad i \le n. \tag{2}$$

We first show that there exist vectors $(x_l^{**})_{l=1}^k$ in $B_{X^{**}} = B_{L_\infty(\mu)}$ such that $x_l^{**}(f_j) = v_{j,l}$ for all $l \le k$, $j \le q$, and $\left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{i,l,s} x_l^{**} \right) - z_{i,s} \right] \right\| \le \delta_i$, for all $i \le n$ and all choices of signs $(\rho_s)_{s=1}^k$. Indeed, let $\sigma_1, \ldots, \sigma_q$ be signs and let $A_1, \dots, A_q$ be distinct atoms in (Ω, Σ, μ) such that $f_j = \sigma_j \chi_{A_j}/\mu(A_j)$ for all $j \leq q$. We can assume without loss of generality that the A_j's are pairwise disjoint. We define $(x_l^{**})_{l=1}^k$ in $L_\infty(\mu)$ as follows:

$$x_l^{**}|A_j = \sigma_j v_{j,l}, \quad j \leq q, \quad \text{while } x_l^{**}|\Omega \setminus \bigcup_{j \leq q} A_j = z_{n,l}|\Omega \setminus \bigcup_{j \leq q} A_j,$$

where we regard $z_{n,l}$ as an element of $X^{**} = L_\infty(\mu)$. Clearly, $\|x_l^{**}\| \leq 1$ and $x_l^{**}(f_j) = \int_{A_j} \sigma_j v_{j,l} \sigma_j \chi_{A_j}/\mu(A_j) d\mu = v_{j,l}$, for all $j \leq q$ and $l \leq k$.

Given signs $\rho_1, \dots, \rho_k$ and $i \leq n$ we have that

$$\begin{aligned} (3) \qquad & \Big\| \sum_{s=1}^k \rho_s \Big[\Big(\sum_{l=1}^k a_{i,l,s} x_l^{**} \Big) - z_{i,s} \Big] |\Omega \setminus \bigcup_{j \leq q} A_j \Big\| \\ & \leq \Big\| \sum_{s=1}^k \rho_s \Big[\Big(\sum_{l=1}^k a_{i,l,s} z_{n,l} \Big) - z_{i,s} \Big] \Big\| < \delta_i \quad \text{by (1).} \end{aligned}$$

We next fix signs $\rho_1, \dots, \rho_k$, $i \leq n$ and $j \leq q$. We set

$$H_{i,j}(t) = \sum_{s=1}^k \rho_s \Big[\Big(\sum_{l=1}^k a_{i,l,s} \sigma_j v_{j,l} \Big) - z_{i,s}(t) \Big], \quad t \in A_j.$$

We claim that $|H_{i,j}| < \delta_i$, μ-almost everywhere in A_j. Indeed, note first that $\int_{A_j} z_{i,s} d\mu = \sigma_j \mu(A_j) f_j(z_{i,s})$ for all $s \leq k$, and thus

$$\begin{aligned} (4) \quad \Big| \int_{A_j} H_{i,j} d\mu \Big| &= \Big| \sum_{s=1}^k \rho_s \Big[\Big(\sum_{l=1}^k a_{i,l,s} \sigma_j v_{j,l} \mu(A_j) \Big) - \sigma_j \mu(A_j) f_j(z_{i,s}) \Big] \Big| \\ &\leq \mu(A_j) \sum_{s=1}^k \Big| \Big(\sum_{l=1}^k a_{i,l,s} v_{j,l} \Big) - f_j(z_{i,s}) \Big| < \delta_i \mu(A_j), \quad \text{by (2).} \end{aligned}$$

On the other hand, setting $B_{i,j} = \{t \in A_j : |H_{i,j}(t)| \geq \delta_i\}$ and taking into account that A_j is an atom, we infer that $\mu(B_{i,j}) = 0$. Indeed, otherwise, $\mu(B_{i,j}) = \mu(A_j)$ and thus $|H_{i,j}| \geq \delta_i$, μ-almost everywhere on A_j. But also, as A_j is an atom, $H_{i,j}$ has a constant sign μ-almost everywhere on A_j and so $|\int_{A_j} H_{i,j} d\mu| \geq \delta_i \mu(A_j)$, contradicting (4). Therefore, $\mu(B_{i,j}) = 0$ and hence $|H_{i,j}| < \delta_i$, μ-almost everywhere in A_j, as claimed. We conclude that

$$\Big\| \sum_{s=1}^k \rho_s \big[\big(\sum_{l=1}^k a_{i,l,s} x_l^{**} \big) - z_{i,s} \big] |A_j \Big\| \leq \delta_i, \quad \text{for all } i \leq n, \ j \leq q,$$

and all choices of signs $(\rho_s)_{s=1}^k$. Combining with (3) we deduce that

$$\Big\|\sum_{s=1}^k \rho_s\big[\big(\sum_{l=1}^k a_{i,l,s}x_l^{**}\big) - z_{i,s}\big]\Big\| \leq \delta_i,$$

for all $i \leq n$ and all choices of signs $(\rho_s)_{s=1}^k$.

We next set $W = [\{x_l^{**} : l \leq k\} \cup \{z_{i,s} : i \leq n, s \leq k\}]$ and choose $0 < \delta < \epsilon$. Theorem 2.1 yields a linear operator $U: W \to X$, $\|U\| \leq 1+\delta$, so that $U|X\cap W = id_{X\cap W}$ and $g(f_j) = f_j(Ug)$, for all $g \in W$ and $j \leq q$. Setting $x_l = Ux_l^{**}/(1+\delta)$, $l \leq k$, we obtain that for every choice of signs $\rho_1, \ldots, \rho_k$ and all $i \leq n$,

$$\begin{aligned}
&\left\|\sum_{s=1}^k \rho_s\left[\left(\sum_{l=1}^k a_{i,l,s}x_l\right) - z_{i,s}\right]\right\| \\
&= \left\|\sum_{s=1}^k \rho_s\left[\left(\sum_{l=1}^k a_{i,l,s}Ux_l^{**}/(1+\delta)\right) - Uz_{i,s}\right]\right\| \\
&\leq (1+\delta)\left\|\sum_{s=1}^k \rho_s\left[\left(\sum_{l=1}^k a_{i,l,s}x_l^{**}/(1+\delta)\right) - z_{i,s}\right]\right\| \\
&\leq \left\|\sum_{s=1}^k \rho_s\left[\left(\sum_{l=1}^k a_{i,l,s}x_l^{**}\right) - (1+\delta)z_{i,s}\right]\right\| \\
&\leq \left\|\sum_{s=1}^k \rho_s\left[\left(\sum_{l=1}^k a_{i,l,s}x_l^{**}\right) - z_{i,s}\right]\right\| + \delta\left\|\sum_{s=1}^k \rho_s z_{i,s}\right\| \leq \delta_i + \delta, \quad \text{as } \|T_i\| \leq 1.
\end{aligned}$$

Thus $\sum_{s=1}^k \big|x^*\big(\sum_{l=1}^k a_{i,l,s}x_l\big) - x^*(z_{i,s})\big| \leq \delta_i + \delta$, for all $x^* \in B_{X^*}$. If we define $S: X^* \to V$ by $Sx^* = \sum_{l=1}^k x^*(x_l)e_l$, we see that S is w^*-continuous and $\|S\| \leq 1$. Indeed, for the latter assertion we observe that for every choice of signs $\rho_1, \ldots, \rho_k$, $\|\sum_{l=1}^k \rho_l x_l^{**}\| \leq 1$, by the definition of the sequence $(x_l^{**})_{l=1}^k$ and the fact that $\|v_j\| \leq 1$ for $j \leq q$, and $\|T_n\| \leq 1$. It follows now, by the choice of U, that $\|\sum_{l=1}^k \rho_l x_l\| \leq 1$ for every choice of signs $\rho_1, \ldots, \rho_k$, and therefore $\|S\| \leq 1$.

We deduce that $\|R_iSx^* - T_ix^*\| \leq \delta_i + \delta$ for all $x^* \in B_{X^*}$ and every $i \leq n$. Hence $\|R_iS - T_i\| < \delta_i + \epsilon$ for all $i \leq n$. Finally,

$$\begin{aligned}
\|Sf_j - v_j\| &= \sum_{l=1}^k |f_j(x_l) - v_{j,l}| \\
&= \sum_{l=1}^k |(1+\delta)^{-1}f_j(Ux_l^{**}) - v_{j,l}| \\
&= \sum_{l=1}^k |(1+\delta)^{-1}x_l^{**}(f_j) - v_{j,l}|
\end{aligned}$$

$$= \delta(1+\delta)^{-1}\|v_j\| < \epsilon.$$

The proof of the proposition is now complete. ∎

Proof of Theorem 3.2: We first choose $\theta_i < \delta_i$ so that $\|R_iT_n - T_i\| < \theta_i$ and $\|R_iv_j - T_if_j\| < \theta_i$ for all $j \leq q$ and $i \leq n$. We then choose $0 < \epsilon_0 < (\delta_i - \theta_i)/4$ for all $i \leq n$, and a sequence (ϵ_m) of positive scalars such that $\sum_{m=1}^{\infty} \epsilon_m < \epsilon_0$.

Proposition 3.3 yields a w^*-continuous linear operator S_0: $X^* \to V$, $\|S_0\| \leq 1$, such that $\|R_iS_0 - T_i\| < \theta_i + \epsilon_0$ for $i \leq n$, and $\|S_0f_j - v_j\| < \epsilon_0$, for $j \leq q$.

We next apply Proposition 3.3 for "n"$= 1$, "R_1"$= id_V$, "T_1"$= S_0$, "δ_1"$= \epsilon_0$ and "ϵ"$=\epsilon_1$, to obtain a w^*-continuous linear operator S_1: $X^* \to V$, $\|S_1\| \leq 1$, so that $\|S_0 - S_1\| < \epsilon_0 + \epsilon_1$ and $\|S_1f_j - v_j\| < \epsilon_1$, for $j \leq q$. Continuing in this fashion we construct w^*-continuous linear operators S_m: $X^* \to V$ with $\|S_m\| \leq 1$ and such that

$$\|S_{m-1} - S_m\| < \epsilon_{m-1} + \epsilon_m, \|S_mf_j - v_j\| < \epsilon_m, \quad \text{for all } j \leq q,\ m \in \mathbb{N}.$$

It is clear that the sequence of operators (S_m) converges in norm to a w^*-continuous linear operator T: $X^* \to V$ such that $\|T\| \leq 1$ and $Tf_j = v_j$, $j \leq q$. In addition to that we have

$$\|S_m - S_0\| < \epsilon_0 + 2\sum_{i=1}^{m-1} \epsilon_i + \epsilon_m, \quad m > 1,$$

and thus $\|T - S_0\| < 3\epsilon_0$. We conclude that

$$\begin{aligned}\|R_iT - T_i\| &\leq \|R_i(T - S_0)\| + \|R_iS_0 - T_i\| \\ &< \theta_i + 4\epsilon_0 < \delta_i,\ i \leq n.\end{aligned}$$ ∎

COROLLARY 3.4: *Let X be an L_1-predual and let V be a subspace of X^* isometric to ℓ_1^k. Assume that there exist collections of vectors $(f_j)_{j=1}^q \subset \mathrm{ext}(B_{X^*})$ and $(v_j)_{j=1}^q \subset B_V$, with $(f_j)_{j=1}^q$ linearly independent, so that every linear operator R: $X^* \to X^*$ satisfying $Rf_j = v_j$ for all $j \leq q$, also satisfies $R|V = id_V$. Then there exists a w^*-continuous contractive projection P: $X^* \to V$, such that $Pf_j = v_j$ for all $j \leq q$.*

Proof: We apply Theorem 3.2 for $n = 1$, $R_1 = id_V$, $T_1 = \mathbf{0}$ and $\delta_1 > 1$, to obtain a w^*-continuous linear operator P: $X^* \to V$, $\|P\| \leq 1$, so that $Pf_j = v_j$ for all $j \leq q$. Our assumptions yield that P is the desired projection. ∎

Remark: We note that Lemma 3.1 and Corollary 3.2 of [10] yield for every $\epsilon > 0$ a w^*-continuous projection $P\colon X^* \to V$, $\|P\| \leq 1+\epsilon$, such that $Pf_j = v_j$ for all $j \leq q$.

4. Main results

In this section we present the proofs of the results mentioned in the introduction.

Proof of Theorem 1.1: Assume K is infinite and let (e_n^*) be an enumeration of K. (The argument for finite K is implicitly contained in the proof of the infinite case.) It is clear that (e_n^*) is isometrically equivalent to the usual ℓ_1-basis. Set $Y_n = [e_1^*, \dots e_n^*]$, $n \in \mathbb{N}$ and let (δ_n) be a null sequence of positive scalars. We shall inductively construct w^*-continuous contractive projections $P_n\colon X^* \to Y_n$ such that $\|P_k P_n - P_k\| < \delta_k$, whenever $k \leq n$. P_1 is selected by applying Corollary 3.4 for the subspace Y_1 and the vectors $f_1 = v_1 = e_1^*$. Suppose $(P_i)_{i=1}^n$ have been selected so that $\|P_i P_j - P_i\| < \delta_i$ whenever $i \leq j \leq n$. Apply Theorem 3.2 for "V"$= Y_{n+1}$, "T_i"$= P_i$, "R_i"$= P_i|Y_{n+1}$, $i \leq n$, and the collections of vectors "$(f_j)_{j=1}^q$"$= (e_j^*)_{j=1}^{n+1}$, "$(v_j)_{j=1}^q$"$= (e_j^*)_{j=1}^{n+1}$, in order to obtain a w^*-continuous linear operator $P_{n+1}\colon X^* \to Y_{n+1}$, $\|P_{n+1}\| \leq 1$, such that $P_{n+1}e_j^* = e_j^*$ for all $j \leq n+1$, and $\|P_i P_{n+1} - P_i\| < \delta_i$ for all $i \leq n$. Clearly, P_{n+1} is the required projection onto Y_{n+1}. This completes the inductive construction. The assertion of the theorem now follows from Proposition 3.1. ∎

Proof of Corollary 1.2: Clearly, K is linearly independent. When K is finite the assertion follows immediately from Theorem 1.1 as $[K]$ is isometric to $\ell_1^{|K|} = C(K)^*$. If K is infinite let (e_n^*) be an enumeration of K and set $Y = [(e_n^*)]$. Of course (e_n^*) is isometrically equivalent to the usual ℓ_1-basis, and applying the Choquet representation and the Krein–Millman theorems, we infer that $\overline{\mathrm{co}}^{w^*}(K \cup -K) = B_Y$. A classical result [12] yields that Y is w^*-closed in X^*. It is not hard to see (cf. also Lemma 2 of [2]) that Y is w^*-isometric to $C(K)^*$. The result follows from Theorem 1.1. ∎

Proof of Corollary 1.3: We regard B_{X^*} in its w^*-topology and set $H = \mathrm{ext}(B_{X^*})$. Since X is separable and X^* is non-separable, H is an uncountable G_δ-subset of B_{X^*}. It follows that H is an uncountable Polish space in its relative w^*-topology. We will show that H contains a w^*-compact subset L homeomorphic to the Cantor set Δ, such that $L \cap (-L) = \emptyset$. Indeed, let $\sigma\colon H \to H$ denote negation ($\sigma h = -h$). Then σ is a fixed-point free homeomorphism on the uncountable Polish space H and therefore there exists an uncountable relatively open subset

G of H such that $G \cap \sigma G = \emptyset$. By a classical result G contains a compact subset L homeomorphic to the Cantor set which clearly satisfies $L \cap (-L) = \emptyset$.

It is clear that L contains homeomorphs of all countable compact metric spaces. Corollary 1.2 now yields that X contains subspaces isometric to $C(K)$, for every countable compact metric space K. Because X is separable, a result of Bourgain (Proposition 9 of [4]) implies that X contains a subspace isometric to $C(\Delta)$. The existence of a contractively complemented subspace of X isometric to $C(\Delta)$ now follows from a result of Pelczynski [22]. ∎

Proof of Theorem 1.4: We first choose an infinite w^*-convergent subsequence $(e^*_m)_{m\in M}$ of (e^*_n) and set $x^*_0 = w^* - \lim_{m\in M} e^*_m$. Clearly, $\|x^*_0\| \leq 1$. If $x^*_0 = \mathbf{0}$ then $Z = [(e^*_m)_{m\in M}]$ is w^*-closed in X^* by Lemma 1 of [2]. We deduce from Theorem 1.1 that Z is the range of a w^*-continuous contractive projection in X^*. It is easy to see that if (r_n) is an enumeration of M then the subspace $Y = [(e^*_{r_{2n}} - e^*_{r_{2n-1}})]$ is the range of a w^*-continuous contractive projection in Z. Hence by composing the projections previously obtained we see that the assertion of the theorem holds in this case.

We shall next deal with the case of $x^*_0 \neq \mathbf{0}$. Suppose that $x^*_0 = \sum_{j=1}^{\infty} a_j e^*_j$ and choose a sequence of positive scalars (ϵ_i) such that $\sum_{i=1}^{\infty} \epsilon_i < 1$. Choose also $n_1 \in \mathbb{N}$ so that $\sum_{j>n_1} |a_j| < \epsilon_1 \sum_{j\leq n_1} |a_j|$. We shall inductively construct increasing sequences $(m_k)_{k=1}^{\infty} \subset M$ and $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ with $n_k < m_{2k-1} < m_{2k} < n_{k+1}$, and w^*-continuous contractive projections P_k: $X^* \to Y_k$, where $Y_k = [u^*_i : i \leq k]$ and $u^*_k = (e^*_{m_{2k}} - e^*_{m_{2k-1}})/2$, $k \in \mathbb{N}$, so that the following conditions are fulfilled:

$$(5) \qquad \sum_{j>n_i} |a_j| < \epsilon_i \sum_{j\leq n_1} |a_j|, \quad i \in \mathbb{N}.$$

$$(6) \qquad P_1 e^*_j = \mathbf{0}, \quad j \leq n_1, \quad \text{while } \|P_i e^*_j\| < \sum_{l<i} \epsilon_l, \quad j \leq n_1, \quad i \geq 2.$$

$$(7) \qquad P_i\Big(\sum_{j\leq n_i} a_j e^*_j\Big) = \mathbf{0}, \quad i \in \mathbb{N}.$$

$$(8) \qquad P_i e^*_{m_{2j}} = u^*_j,\ P_i e^*_{m_{2j-1}} = -u^*_j, \quad j \leq i, \quad i \in \mathbb{N}.$$

$$(9) \qquad \|P_i P_j - P_i\| < \sum_{l=i}^{j-1} \epsilon_l, \quad i < j \text{ in } \mathbb{N}.$$

$$(10) \qquad \|P_i e^*_{m_j}\| < \epsilon_i, \quad j \in \{2l-1, 2l\}, \quad i < l \text{ in } \mathbb{N}.$$

Once this is accomplished, condition (9) will enable us to apply Proposition 3.1 and deduce that $Y = [(u^*_k)]$ is the range of a w^*-continuous projection in

X^*. Note that Y is w^*-closed in X^* by Lemma 1 of [2] as (u_k^*) is isometrically equivalent to the usual ℓ_1-basis.

We first choose $m_1 < m_2$ in M with $m_1 > n_1$, and apply Corollary 3.4 for "V"$= Y_1$, "q"$= n_1+2$, "f_j"$= e_j^*$ $(j \leq n_1)$, "f_{q-1}"$= e_{m_1}^*$, "f_q"$= e_{m_2}^*$, and "v_j"$= \mathbf{0}$ $(j \leq n_1)$, "v_{q-1}"$= -u_1^*$, "v_q"$= u_1^*$. We obtain a w^*-continuous contractive projection P_1: $X^* \to Y_1$ such that $P_1 e_j^* = \mathbf{0}$ for all $j \leq n_1$.

Suppose that we have constructed $(n_i)_{i=1}^k$, $(m_i)_{i=1}^{2k}$ and $(P_i)_{i=1}^k$ so that conditions (5)–(10) are satisfied. We next choose $n_{k+1} > m_{2k}$ so that (5) is satisfied for $i = k+1$. By (5) and (7) of the induction hypothesis we infer that $\|P_i x_0^*\| < \epsilon_i$, for $i \leq k$. We can therefore choose $m_{2k+1} < m_{2k+2}$ in M with $m_{2k+1} > n_{k+1}$, such that $\|P_i e_{m_{2k+1}}^*\| < \epsilon_i$ and $\|P_i e_{m_{2k+2}}^*\| < \epsilon_i$ for all $i \leq k$. Hence (10) is satisfied for $l = k+1$.

We next put $q = n_{k+1} + 2$ and set $f_j = e_j^*$, for $j \leq n_{k+1}$, $f_{q-1} = e_{m_{2k+1}}^*$ and $f_q = e_{m_{2k+2}}^*$. We claim that there exist vectors $(v_j)_{j=1}^q$ in $B_{Y_{k+1}}$ so that

$$\|v_j\| < \sum_{l=1}^{k} \epsilon_l, \quad j \leq n_1. \tag{11}$$

$$v_j = P_k e_j^*, \quad j \in (n_1, n_{k+1}], \quad v_{q-1} = -u_{k+1}^*, \quad v_q = u_{k+1}^*. \tag{12}$$

$$\sum_{j \leq n_{k+1}} a_j v_j = \mathbf{0}. \tag{13}$$

$$\|P_i f_j - P_i v_j\| < \sum_{l=i}^{k} \epsilon_l, \quad i \leq k, \quad j \leq q. \tag{14}$$

Having achieved this and taking into account (9) of the induction hypothesis, we employ Theorem 3.2 for "V"$= Y_{k+1}$, "n"$= k$, "δ_i"$= \sum_{l=i}^k \epsilon_l$, "$T_i$"$= P_i$, "$R_i$"$= P_i | Y_{k+1}$ $(i \leq k)$, and the collections of vectors $(f_j)_{j=1}^q$, $(v_j)_{j=1}^q$ described above, to find a w^*-continuous linear operator P_{k+1}: $X^* \to Y_{k+1}$, $\|P_{k+1}\| \leq 1$, such that $\|P_i P_{k+1} - P_i\| < \sum_{l=i}^k \epsilon_l$, for all $i \leq k$, and $P_{k+1} f_j = v_j$ for all $j \leq q$. It is easy to verify that P_{k+1} is a projection onto Y_{k+1} so that $(n_i)_{i=1}^{k+1}$, $(m_i)_{i=1}^{2k+2}$ and $(P_i)_{i=1}^{k+1}$ satisfy conditions (5)–(10).

The collection $(v_j)_{j=n_1+1}^q$ is explicitly defined in (12). It remains to define $(v_j)_{j=1}^{n_1}$. We first choose scalars $(b_{i,l})$, where $i \leq k$ and $l \in (n_k, n_{k+1}]$, such that $P_k e_l^* = \sum_{i=1}^k b_{i,l} u_i^*$. Note that $\sum_{i=1}^k |b_{i,l}| \leq 1$ for every $l \in (n_k, n_{k+1}]$ since $\|P_k\| = 1$. We also define scalars $(\rho_i)_{i=1}^k$ by

$$\rho_i = \Big(- \sum_{l \in (n_k, n_{k+1}]} a_l b_{i,l} \Big) \Big/ \sum_{j \leq n_1} |a_j|,$$

and set $\rho_{i,j} = sg(a_j)\rho_i$ for $i \le k$ and $j \le n_1$. Observe that $\sum_{i=1}^{k} |\rho_i| < \epsilon_k$, by (5) of the induction hypothesis. We now set

$$v_j = P_k e_j^* + \sum_{i=1}^{k} \rho_{i,j} u_i^*, \quad j \le n_1.$$

It follows now by (6) of the induction hypothesis that (11) is satisfied. To establish (13) we have

$$\begin{aligned}
\sum_{j\le n_{k+1}} a_j v_j &= \sum_{j \le n_1} a_j v_j + \sum_{j\in(n_1,n_{k+1}]} a_j P_k e_j^* \\
&= \sum_{j\le n_1} a_j \sum_{i=1}^{k} \rho_{i,j} u_i^* + \sum_{j\le n_{k+1}} a_j P_k e_j^* \\
&= \sum_{j\le n_1} |a_j| \sum_{i=1}^{k} \rho_i u_i^* + \sum_{j \le n_k} a_j P_k e_j^* + \sum_{j\in(n_k,n_{k+1}]} a_j P_k e_j^* \\
&= -\sum_{i=1}^{k} \sum_{l\in(n_k,n_{k+1}]} a_l b_{i,l} u_i^* + \sum_{j\in(n_k,n_{k+1}]} a_j P_k e_j^*, \quad \text{by (7)} \\
&= - \sum_{l\in(n_k,n_{k+1}]} a_l \sum_{i=1}^{k} b_{i,l} u_i^* + \sum_{j\in(n_k,n_{k+1}]} a_j P_k e_j^* \\
&= - \sum_{l\in(n_k,n_{k+1}]} a_l P_k e_l^* + \sum_{j\in(n_k,n_{k+1}]} a_j P_k e_j^* = \mathbf{0}.
\end{aligned}$$

Finally, we show that (14) holds. Indeed, when $j \in \{q-1, q\}$, this is a consequence of the choice of m_{2k+1} and m_{2k+2}. When $j \in (n_1, n_{k+1}]$ the assertion follows from (9) of the induction hypothesis. When $j \le n_1$ it follows from (9) of the induction hypothesis and the fact that $\sum_{i=1}^{k} |\rho_i| < \epsilon_k$. ∎

Proof of Corollary 1.5:] If X^* is non-separable the assertion follows from Corollary 1.3. If X^* is separable, then $\mathrm{ext}(B_{X^*})$ is countable and X^* is isometric to ℓ_1. Let (e_n^*) be a basis for X^* isometrically equivalent to the usual ℓ_1-basis, and choose a w^*-convergent subsequence $(e_{m_n}^*)$ of (e_n^*) according to Theorem 1.4. Let $Y = [(e_{m_{2n}}^* - e_{m_{2n-1}}^*)]$. Then it is easy to see that Y is w^*-isometric to c_0^*. The result follows from Theorem 1.4. ∎

Our last corollary is a special case of the structural result for separable L_1-preduals established in [20] and [15].

COROLLARY 4.1: *Suppose that X^* is isometric to ℓ_1. Then there exists a sequence (E_n) of finite-dimensional subspaces of X such that $E_n \subset E_{n+1}$ for all n, each E_n is isometric to ℓ_∞^n, and $\bigcup_{n=1}^\infty E_n$ is dense in X.*

Proof: Let (e_n^*) be a basis for X^* isometrically equivalent to the usual ℓ_1-basis. Let $Y_n = [e_i^* : i \leq n]$, $n \in \mathbb{N}$, and let (δ_n) be a sequence of positive scalars such that $\sum_{n=1}^\infty \delta_n < \infty$. The argument in the proof of Theorem 1.1 now yields a sequence (P_n) of w^*-continuous contractive projections in X^* with $\operatorname{Im} P_n = Y_n$ and such that $\|P_k P_n - P_k\| < \delta_k$ whenever $k \leq n$. Given $k \leq n$ in $\mathbb{N}$, we set $Q_k^n = P_k \cdots P_n$. Clearly Q_k^n is a w^*-continuous contractive projection onto Y_k. Moreover, our assumptions yield that $\|Q_k^n - Q_k^{n+1}\| < \delta_n$, for all $n \geq k$ and thus the sequence of operators $(Q_k^n)_{n \geq k}$ converges in norm to a w^*-continuous contractive projection Q_k from X^* onto Y_k. It is easily seen that $Q_k^m Q_l^n = Q_k^n$ whenever $k \leq l \leq m \leq n$ and hence $Q_k Q_l = Q_k$ when $k \leq l$.

We now let $E_n = Q_n^* Y_n^*$. E_n is naturally identified to a subspace of X as Q_n is w^*-continuous, and of course it is isometric to ℓ_∞^n for all $n \in \mathbb{N}$. Since $Q_n Q_{n+1} = Q_n$ we deduce that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$. It is also easily verified that Q_n^* acts as a contractive projection from X onto E_n. Rainwater's theorem now yields that $\lim_n Q_n^* x = x$, weakly, for all $x \in X$ and thus $\bigcup_{n=1}^\infty E_n$ is dense in X. ∎

Remark:

1. We note that the proof of Corollary 1.5 that appears in [27] makes use of the structural result established in [20] and [15].
2. Theorem 1.4 is no longer valid if we consider isomorphic ℓ_1-preduals. Indeed, it is shown in [5] that there exist isomorphic ℓ_1-preduals which do not contain isomorphic copies of c_0.
3. According to a result of Fonf [7], every Banach space X such that $\operatorname{ext}(B_{X^*})$ is countable contains a subspace isomorphic to c_0.

It was shown in [11] that every separable L_1-predual is isometric to a quotient of $C(\Delta)$. It is an open problem [2] whether every ℓ_1-predual is isomorphic, or even almost isometric, to a quotient of $C(K)$ for some countable compact metric space K.

Question: Suppose X is an ℓ_1-predual such that for some $\epsilon > 0$ and some countable ordinal α, the ϵ-Szlenk index of X [25] exceeds ω^α. Does X contain a contractively complemented subspace isometric to $C_0(\omega^{\omega^\alpha})$? Does X contain a subspace isomorphic to $C(\omega^{\omega^\alpha})$?

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